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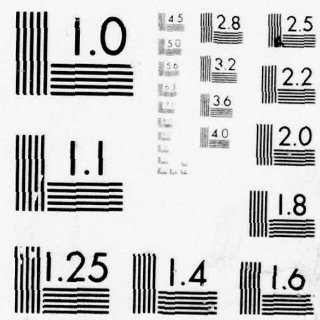

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THEORY AND APPLICATIONS OF GENERALIZED WALSH FUNCTIONS IN DIGITAL SYSTEMS

by

David K. Cheng

Final Technical Report

Grant No. AFOSR-75-2809
Project-Task No. 2304/A3
1 March 1975 to 30 April 1979

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Syracuse University
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is a final technical report ~~for Grant No.~~ AFOSR-75-2809 on the theory and applications of generalized Walsh functions in digital systems. Research accomplishments in the areas of ~~(1)~~ Walsh-Transform solution of difference equations; ~~(2)~~ development of an algorithm for sequency ordering of Hadamard functions; ~~(3)~~ development of a general two-parameter orthogonal transformation matrix; and ~~(4)~~ analysis of the stochastic behavior of digital combinational circuits are summarized. Reprints are attached.

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TABLE OF CONTENTS

I. INTRODUCTION

Walsh functions[1-5] are binary-valued, periodic functions which form a complete orthogonal set on the interval [0,1). Because their pulse-like character is compatible with the operation of digital computers and processors, much recent interest has been generated in the study of the properties and applications of Walsh functions. The study of the characteristics and capabilities of Walsh functions at Syracuse University began in 1972. Past accomplishments include the formulation of a sampling theorem for sequency-limited waveforms,[6] Walsh-transform analysis of discrete dyadic-invariant systems,[7] time-domain analysis of dyadic-invariant systems,[8] a critical study of the radiation characteristics of a linear array of short dipoles excited by Walsh-shaped currents,[9] and a clarification of the interrelationships and transconversion procedures of Paley, Hadamard, and Walsh functions.[10]

The emphasis of the present grant was directed toward developing some particularly useful applications of Walsh functions and Walsh transforms. Many possible areas of application were examined. This report summarizes the more significant results obtained under this grant. Three new areas of application have been identified. They are: solution of difference equations; multidimensional signal processing on a real-time basis; and noise-error determination of combinational circuits. In addition, a simple algorithm for obtaining the sequency vector of high-order Hadamard transform matrices has been developed. These research accomplishments, together with a list of publications, a list of professional personnel, and the reprints of published papers, comprise this final technical report.

II. RESEARCH ACCOMPLISHMENTS

In this section we summarize the more significant research results obtained under this grant.

1. Walsh-transform solution of difference equations - Dyadic operations are fundamental to Walsh functions and their applications. However, dyadic-invariant systems, though mathematically and conceptually interesting, do not correspond to real-world physical systems. In order for Walsh functions and Walsh transforms to be more useful and more effective in signal processing and other applications, methods for their usage in analyzing nondyadic time-invariant systems must be found. It is essential that the Walsh transforms for time-shifted functions be related to those before the shift so that important operations such as time-delay simulation, convolution, and correlation can be dealt with. We have obtained a relation between the Walsh transforms of a function subject to dyadic and nondyadic time displacements. By defining spectrum-conversion matrices, we have demonstrated the ease of using Walsh transformation to solve linear difference equations.

2. Development of an algorithm for sequency ordering of Hadamard functions - Hadamard functions, like Walsh functions, are binary-valued functions which form a complete orthonormal set on [0,1). However, unlike Walsh functions, the order h of a Hadamard function H(h,t) is not equal to its sequency, which is determined from the inverse Gray code of the bit-reverse binary representation of h[10]. In signal-

processing work it is desirable that the rows of a Hadamard matrix be rearranged in increasing sequency, resulting in a Walsh matrix. Recursive relations for sequency vectors of consecutive orders have been obtained, from which an algorithm can be devised to generate the sequency vector of a Hadamard matrix.

3, Development of a general two-parameter orthogonal transformation matrix -- A procedure frequently used in two-dimensional image processing is to scan the image sequentially in time, converting it into a one-dimensional digital signal and then processing it in blocks in the transform domain. However, this conversion alters the relative positions of the image elements and hence changes the character of the signal. We have introduced a two-parameter generalized transformation matrix which reduces to the Fourier and Hadamard matrices under special conditions. It has been shown that the new trnasformation matrix with appropriate parameters will preserve the proper relationship in the transform domain before and after scanning.

4. Analysis of the stochastic behavior of digital combinational circuits -- The determination of the output behavior of digital systems in response to stochastic inputs has been a formidable task because it cannot be handled by the usual linear algebra and calculus. Logical algebra and an associated calculus that is digital in nature are required. We have found Walsh functions to be relevant in this connection. An n-input Boolean function is expressed as a Walsh series which

facilitates the analysis of the statistical error at the output of a digital combinational circuit due to a signal corrupted by noise. We think this work represents a very significant contribution to the analysis of the stochastic behavior of digital combinational circuits. Further work on this technique will undoubtedly lead to new results for sequential circuits.

III. LIST OF PUBLICATIONS

Besides the technical report on "Paley, Hadamard, and Walsh Functions: Interrelationships and Transconversions,"[10] the following articles relevant to this grant were published during the epriod covered by this final report.

1. "Time-Shift Theorems for Walsh Transforms and Solution of Difference Equations," by D.K. Cheng and J.J. Liu, IEEE Transactions on Electromagnetic Compatibility, vol. EMC-18, pp. 83-87, May 1976.

   Abstract - Several time-shift theorems for Walsh transforms of functions subject to nondyadic as well as dyadic time displacements are presented. Spectrum-conversion matrices are defined and a relation between a function with an ordinary shift and that with a dyadic shift is established. Procedures for solving difference equations by Walsh transformation are given.

2. "An Algorithm for Sequency Ordering of Hadamard Functions," by D.K. Cheng and J.J. Liu, IEEE Transactions on Computers, vol. C-26, pp. 308-309, March 1977.

   Abstract - A simple algorithm is developed for obtaining the sequency vector of high-order Hadamard transform matrices without the need for converting the order of individual Hadamard functions to sequency.

3. "A Generalized Orthogonal Transformation Matrix," by D.K. Cheng and J.J. Liu, IEEE Transactions on Computers, vol. C-28, pp. 147-150, February 1979.

   Abstract - A procedure is described for generating a two-parameter orthogonal transformation matrix which reduces to the Fourier and Hadamard transformation matrices under special conditions. This generalized transformation matrix is particularly useful for multidimensional signal processing on a real-time basis because it preserves a proper relationship in the transform domain.

4. "Noise-Error Determination of Combinational Circuits by Walsh Functions," by A.U. Shankar and D.K. Cheng," IEEE Transactions on Electromagnetic Compatibility, vol. EMC-21, pp. 146-152, May 1979.

   Abstract - The stochastic behavior of digital combinational circuits is analyzed by the use of Walsh functions. An n-input Boolean function is represented as a Walsh series and the error caused by noise is measured in terms of a distance which is the fraction of time that the system output due to noise-corrupted signal differs from that due to signal alone. It is shown that the error can be expressed as the sum of two parts: one part depends only on noise statistics, and the other on both signal and noise. Some interesting properties of both parts are discussed and typical examples are given.

IV. LIST OF PROFESSIONAL PERSONNEL

The following professional people participated in the research effort of this grant.

Professor David K. Cheng, Principal Investigator

James J. Liu, Graduate Assistant

Ph.D., January 1977
Dissertation title: "Generalized Walsh Functions - Theory and Applications in Digital Systems."

A. Udaya Shankar, Graduate Assistant

M.S.E.E., August 1978
Thesis title: "Combinational-Circuit Representation and Noise-Error Determination by Walsh Functions."

Y.K. Chen, Graduate Assistant.

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8. D.K. Cheng and J.J. Liu, "Time-domain analysis of dyadic-invariant systems," Proceedings of the IEEE, vol. 62, pp. 1038-1040, July 1974.

9. D.K. Cheng, Characteristics and capabilities of Walsh functions," Final Scientific Report, Grant No. AFOSR-72-2355, Department of Electrical and Computer Engineering, Syracuse University, Syracuse, N.Y., August 1974.

10. D.K. Cheng and J.J. Liu, "Paley, Hadamard and Walsh Functions: Interrelationships and Transconversions," Technical Report No. TR-75-5, Department of Electrical and Computer Engineering, Syracuse University, Syracuse, NY, may 1975.

VI. REPRINTS OF PUBLISHED PAPERS

Time-Shift Theorems for Walsh Transforms and Solution of Difference Equations

DAVID K. CHENG AND JAMES J. LIU

***Abstract*—Several time-shift theorems for Walsh transforms of functions subject to nondyadic as well as dyadic time displacements are presented. Spectrum-conversion matrices are defined and a relation between a function with an ordinary shift and that with a dyadic shift is established. Procedures for solving difference equations by Walsh transformation are given.**

INTRODUCTION

It is well known that a square-integrable function or a periodic function can be approximated in the least integrated-squared-error sense by a linear combination of orthogonal basis functions. When the set of basis functions is "complete," the original function can be approximated in-the-mean with an arbitrarily small error by a sufficiently large number of terms. The most commonly used complete orthogonal basis functions in science and engineering are, without a doubt, sinusoidal functions. They lead to Fourier series and Fourier integrals which are of fundamental importance in signal representation, system analysis, and solution of physical problems.

Sinusoidal functions are, of course, not the only class of possible basis functions. One other class that has received considerable attention recently is the set of Walsh functions [1]–[3]. Walsh functions are binary-valued periodic functions which form a complete orthonormal set on the interval [0,1). Because their pulse-like character is compatible with the operation of digital computers and processers, they have found many important applications in such areas as pattern recognition, signal and image processing, digital filtering, sequency multiplexing, interferometric spectroscopy, etc. [4]–[7]. Generalized transform theories using Walsh functions as the kernels have been developed [8], [9]. Pichler [10], [11] has considered the formalism for the use of Walsh–Fourier transforms in linear system theory. As the main advantage of using Walsh functions lies in their binary character, and as continuous Walsh transforms are difficult and cumbersome to evaluate, Cheng and Liu [12] applied discrete Walsh transforms to the analysis of dyadic-invariant linear systems. However, dyadic-invariant systems, though mathematically and conceptually interesting, do not correspond to ordinary real-world physical systems. In order for Walsh functions and Walsh transforms to be more useful and more effective in signal processing and other applications, methods for their usage in analyzing *nondyadic* time-invariant

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systems must be found. It is essential that the Walsh transforms for time-shifted functions be related to those before the shift so that important operations such as time-delay simulation, convolution, and correlation can be properly dealt with.

This paper presents several time-shift theorems for Walsh transforms of functions subject to nondyadic as well as dyadic time displacements. Spectrum-conversion matrices are defined and a relation between a function with an ordinary shift and that with a dyadic shift is established. As a specific application, the recursive formula relating the output to the input of a system described by a general difference equation is obtained, and a numerical example is included.

Time shifting in the Walsh domain was studied by Brown [13], who made use of the Hadamard matrix instead of the Walsh matrix. The emphasis in Brown's work was mainly on Walsh-domain applications of some signal-processing problems. Solution of difference equations was not considered.

Walsh Transform of Time-Shifted Functions

In this work we confine our attention to sampled time functions and their discrete Walsh spectra. Sampled time functions are represented by column vectors. Their discrete Walsh spectra in the sequency domain, also represented by column vectors, are obtained by premultiplication with a square Walsh matrix of the proper order. Thus let $\bar{f}(i)$ denote the $N \times 1$ $(N = 2^m)$ time vector, then its Walsh spectrum or discrete Walsh transform, $\bar{F}(l)$, is

$$\bar{F} = \bar{\bar{W}}\bar{f} \tag{1}$$

where $\bar{\bar{W}}$ is the $N \times N$ Walsh matrix whose rows and columns are Walsh vectors ordered in an increasing sequency, and hence is different from the Hadamard matrix. The inverse relation of (1) is

$$\bar{f} = \bar{\bar{W}}^{-1}\bar{F} = \frac{1}{N}\bar{\bar{W}}\bar{F}. \tag{2}$$

The ith element of $\bar{f}$ can be written in a summation form

$$f(i) = \frac{1}{N}\sum_{l=0}^{N-1} F(l)W(l,i). \tag{3}$$

A shift of k units in $\bar{f}$ will change the ith element to $f(i-k)$, or

$$f_{-k}(i) = f(i-k) = \frac{1}{N}\sum_{l=0}^{N-1} F(l)W(l, i-k). \tag{4}$$

In matrix form, (4) leads to the sampled shifted function

$$\bar{f}_{-k} = \frac{1}{N}(\bar{\bar{W}}_{-k}\bar{F}) \tag{5}$$

where $\bar{\bar{W}}_{-k}$ is the Walsh matrix $\bar{\bar{W}}$ with its last k rows moved up as the new first k rows and the original first $N-k$ rows each moved down k units. $\bar{\bar{W}}_{-k}$ can also be written as a square matrix whose columns are the Walsh vectors with sequencies from 0 to $N-1$, each shifted by k units:

$$\bar{\bar{W}}_{-k} = [\bar{W}_{-k}(0), \bar{W}_{-k}(1), \cdots, \bar{W}_{-k}(l), \cdots, \bar{W}_{-k}(N-1)]. \tag{6}$$

Let $\bar{F}_{-k}$ be the Walsh spectrum of $\bar{f}_{-k}$; that is,

$$\bar{F}_{-k} = \bar{\bar{W}}\bar{f}_{-k}. \tag{7}$$

Then, from (5), we have

$$\bar{F}_{-k} = \frac{1}{N}\bar{\bar{W}}\bar{\bar{W}}_{-k}\bar{F} = \bar{\bar{B}}_{-k}\bar{F} \tag{8}$$

where

$$\bar{\bar{B}}_{-k} = \frac{1}{N}\bar{\bar{W}}\bar{\bar{W}}_{-k} = \bar{\bar{W}}^{-1}\bar{\bar{W}}_{-k} \tag{9}$$

can be called the *spectrum-conversion matrix* for a k-unit shift, which relates the Walsh spectrum of a sampled function shifted (nondyadically) by k units to that of the sampled function before the shift. Equation (8) can be stated in the form of a theorem.

Theorem 1

If $\bar{F}$ is the discrete Walsh transform of a sampled function $\bar{f}$, then the discrete Walsh transform of the sampled function shifted by k units, $\bar{f}_{-k}$, is $\bar{F}_{-k} = \bar{\bar{B}}_{-k}\bar{F}$, where $\bar{\bar{B}}_{-k} = \bar{\bar{W}}^{-1}\bar{\bar{W}}_{-k}$.

For convenience, let $\bar{\bar{A}}$ be the spectrum-conversion matrix for a 1-unit shift; that is,

$$\bar{\bar{A}} = \bar{\bar{B}}_{-1} = \bar{\bar{W}}^{-1}\bar{\bar{W}}_{-1}. \tag{10}$$

We may write

$$\begin{aligned}\bar{\bar{A}} &= \bar{\bar{W}}^{-1}\bar{\bar{W}}_{-1}(\bar{\bar{W}}^{-1}\bar{\bar{W}}) = \bar{\bar{W}}^{-1}(\bar{\bar{W}}_{-1}\bar{\bar{W}}^{-1})\bar{\bar{W}}\\ &= \bar{\bar{W}}^{-1}\bar{\bar{I}}_{-1}\bar{\bar{W}}\end{aligned} \tag{11}$$

where

$$\bar{\bar{I}}_{-1} = \bar{\bar{W}}_{-1}\bar{\bar{W}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & & 0 & 0\\ 0 & & 1 & & 0\\ \vdots & 0 & & \ddots & \vdots\\ 0 & & & 1 & 0\end{bmatrix} \tag{12}$$

is a unit-shift permutation matrix which is an identity matrix with the last row moved up as the new first row and all other rows shifted down one unit. It is clear that $\bar{\bar{I}}_{-1}$ raised to the kth power will effect a k-unit shift. Hence,

$$(\bar{\bar{I}}_{-1})^k = \bar{\bar{I}}_{-k} = \bar{\bar{W}}_{-k}\bar{\bar{W}}^{-1} \tag{13}$$

and, from (9), we have

$$\begin{aligned}\bar{\bar{B}}_{-k} &= \bar{\bar{W}}^{-1}\bar{\bar{W}}_{-k} = \bar{\bar{W}}^{-1}(\bar{\bar{W}}_{-k}\bar{\bar{W}}^{-1})\bar{\bar{W}}\\ &= \bar{\bar{W}}^{-1}\bar{\bar{I}}^{k}_{-1}\bar{\bar{W}}\\ &= (\bar{\bar{W}}^{-1}\bar{\bar{I}}_{-1}\bar{\bar{W}})(\bar{\bar{W}}^{-1}\bar{\bar{I}}_{-1}\bar{\bar{W}})\cdots(\bar{\bar{W}}^{-1}\bar{\bar{I}}_{-1}\bar{\bar{W}})\\ &= \bar{\bar{A}}^k.\end{aligned} \tag{14}$$

Equation (14) enables us to state the following theorem.

Theorem 2

The spectrum-conversion matrix for a k-unit shift is equal to the kth power of the spectrum-conversion matrix for a 1-unit shift.

We now illustrate the use of the spectrum-conversion matrix for finding the Walsh transform of a shifted sampled time function by an example.

Example 1: A sampled time function is represented by the column vector

$$\bar{f}(i) = \begin{bmatrix}1\\4\\5\\2\\4\\3\\1\\1\end{bmatrix}. \tag{15}$$

Determine the Walsh transform of the shifted time function $\bar{f}(i-1)$.

Solution: We can find the desired discrete Walsh transform $\bar{F}_{-1}$ in two ways.

a) By $\bar{F}_{-1} = \bar{\bar{W}}\bar{f}_{-1}$

$$\bar{F}_{-1} = \bar{\bar{W}}\bar{f}_{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \\ 5 \\ 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 1 \\ -9 \\ -5 \\ -3 \\ 5 \\ -1 \\ -1 \end{bmatrix}. \tag{16}$$

b) By $\bar{F}_{-1} = \bar{\bar{A}}\bar{F}$, where $\bar{\bar{A}}$ is obtained from (11) and $\bar{F} = \bar{\bar{W}}\bar{f}$

$$\bar{F}_{-1} = \bar{\bar{A}}\bar{F}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 21 \\ 3 \\ -7 \\ 3 \\ -5 \\ -7 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 1 \\ -9 \\ -5 \\ -3 \\ 5 \\ -1 \\ -1 \end{bmatrix}. \tag{17}$$

Of course the results in (16) and (17) are the same. Comparing the two column vectors in (17), it is obvious that $\bar{F}_{-1}$ cannot be obtained from $\bar{F}$ by a simple rotation of elements.

Walsh Transform of Dyadically Shifted Functions

From (1), the lth element of $\bar{F}$ can be written in a summation form

$$F(l) = \sum_{i=0}^{N-1} W(l,i)f(i). \tag{18}$$

The lth element of the Walsh transform of $\bar{f}$ dyadically shifted by j units is

$$\begin{aligned} F_{\oplus j}(l) &= \sum_{i=0}^{N-1} W(l,i)f(i \oplus j) \\ &= \sum_{i=0}^{N-1} W(l, i \oplus j)f(i) \\ &= \sum_{i=0}^{N-1} W(l,i)f(i)W(l,j). \end{aligned} \tag{19}$$

In a matrix form, (19) leads to

$$\begin{aligned} \bar{F}_{\oplus j} &= \bar{\bar{C}}_j\bar{\bar{W}}\bar{f} \\ &= \bar{\bar{C}}_j\bar{F} \end{aligned} \tag{20}$$

where $\bar{\bar{C}}_j$ is a diagonal square matrix with elements $W(j,l)$, $l = 0,1,\cdots, N-1$ in the diagonal

$$\bar{\bar{C}}_j = \begin{bmatrix} W(j,0) & & & \bigcirc \\ & W(j,1) & & \\ & & \ddots & \\ \bigcirc & & & W(j, N-1) \end{bmatrix}. \tag{21}$$

Analogously to $\bar{\bar{B}}_{-k}$ in (8), $\bar{\bar{C}}_j$ is the spectrum-conversion matrix for a j-unit dyadic shift, which relates the Walsh transform of a sampled function shifted dyadically by j units to that of the sampled function before the shift.

Theorem 3

If $\bar{F}$ is the discrete Walsh transform of a sampled function $\bar{f}$, then the discrete Walsh transform of the sampled function shifted dyadically by j units, $\bar{f}_{\oplus j}$, is $\bar{F}_{\oplus j} = \bar{\bar{C}}_j\bar{F}$, where $\bar{\bar{C}}_j$ is a diagonal matrix with elements $W(j,l)$, $l = 0,1,\cdots, N-1$ in the diagonal.

It is clear that the matrix $\bar{\bar{C}}_j$ possesses properties similar to those of Walsh functions. In particular,

$$\bar{\bar{C}}_j\bar{\bar{C}}_k = \bar{\bar{C}}_{j\oplus k} \tag{22}$$

and there does not appear to be a simple way to derive $\bar{\bar{C}}_j$ from $\bar{\bar{C}}_1$.

Example 2: For the sampled time function $\bar{f}(i)$ given in (15) find the Walsh transform of the dyadically shifted time function $\bar{f}(i \oplus 1)$.

Solution: We can obtain $\bar{F}_{\oplus 1}$ either from $\bar{\bar{W}}\bar{f}_{\oplus 1}$ or from $\bar{\bar{C}}_1\bar{F}$.

a) From $\bar{\bar{W}}\bar{f}_{\oplus 1}$. We first derive $\bar{f}(i \oplus 1) = \bar{f}_{\oplus 1}$ from the given $\bar{f}(i)$ in (15)

$$\bar{f}_{\oplus 1} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 5 \\ 3 \\ 4 \\ 1 \\ 1 \end{bmatrix}. \tag{23}$$

With (23) we find

$$\bar{F}_{\oplus 1} = \bar{\bar{W}}\bar{f}_{\oplus 1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \\ 5 \\ 3 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 3 \\ -7 \\ 3 \\ 5 \\ 7 \\ 1 \\ -1 \end{bmatrix}. \tag{24}$$

b) From $\bar{\bar{C}}_1\bar{F}$. Here we need to know $\bar{F} = \bar{\bar{W}}\bar{f}$ which has been found as the first column matrix in (17). We have

$$\bar{F}_{\oplus 1} = \bar{\bar{C}}_1\bar{F}$$

$$= \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{bmatrix} \begin{bmatrix} 21 \\ 3 \\ -7 \\ 3 \\ -5 \\ -7 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 3 \\ -7 \\ 3 \\ 5 \\ 7 \\ 1 \\ -1 \end{bmatrix} \tag{25}$$

where $\bar{\bar{C}}_1$ is obtained from (21).

Relations Between Dyadic Shift and Ordinary Shift

By combining (8) and (20), a relation between the Walsh spectrum of a dyadically shifted function and that of a non-dyadically shifted function can be easily established

$$\bar{F}_{\oplus k} = \bar{\bar{D}}_k\bar{F}_{-k} \tag{26}$$

where

$$\bar{\bar{D}}_k = \bar{\bar{C}}_k\bar{\bar{B}}_{-k}{}^{-1} = \bar{\bar{C}}_k\bar{\bar{A}}^{-k}. \tag{27}$$

The inverse Walsh transform of (26) then relates a dyadically shifted sampled function with the sampled function shifted in the conventional manner by the same number of units

$$\bar{f}_{\oplus k} = \bar{\bar{W}}^{-1}\bar{\bar{D}}_k\bar{\bar{W}}\bar{f}_{-k}. \tag{28}$$

Equations (26) and (28) will prove to be useful in Walsh transform analysis of linear systems. The inverse relation of (28) follows directly:

$$\bar{f}_{-k} = \bar{\bar{W}}^{-1}\bar{\bar{D}}_k{}^{-1}\bar{\bar{W}}\bar{f}_{\oplus k}. \tag{29}$$

Solution of Difference Equations

With the aid of theorems 1 and 2, we can determine the solution of a difference equation by Walsh transformation in a straightforward manner. Moreover, we can obtain the recursive formula of the system governed by the difference equation. Both procedures are illustrated in the following example.

Example 3

Solve the following difference equation:

$$2y(n-2) + 5y(n-1) + 2y(n) = x(n). \tag{30}$$

Solution: a) To solve (30), we write it in vector form using column vectors to represent discrete sampled functions

$$2\bar{y}_{-2} + 5\bar{y}_{-1} + 2\bar{y} = \bar{x}. \tag{31}$$

If $\bar{Y}$ and $\bar{X}$ denote the discrete Walsh transforms of $\bar{y}$ and $\bar{x}$, respectively, then the Walsh transformation of (31) yields, in view of theorems 1 and 2,

$$(2\bar{\bar{A}}^2 + 5\bar{\bar{A}} + 2\bar{\bar{I}})\bar{Y} = \bar{X} \tag{32}$$

where $\bar{\bar{A}}$ has been given in (11). Consequently,

$$\bar{Y} = (2\bar{\bar{A}}^2 + 5\bar{\bar{A}} + 2\bar{\bar{I}})^{-1}\bar{X} = \bar{\bar{H}}\bar{X} \tag{33}$$

with

$$\bar{\bar{H}} = (2\bar{\bar{A}}^2 + 5\bar{\bar{A}} + 2\bar{\bar{I}})^{-1} \tag{34}$$

which is effectively the transfer function in the sequency domain. The solution of the difference equation is then

$$\bar{y} = \bar{\bar{W}}^{-1}\bar{Y} = \bar{\bar{W}}^{-1}\bar{\bar{H}}\bar{X}$$

$$= \bar{\bar{W}}^{-1}\bar{\bar{H}}\bar{\bar{W}}\bar{x}. \tag{35}$$

b) To obtain a recursive formula in the following form

$$y(n) = a_0x(n) + a_1x(n-1) + \cdots + a_{N-1}x(n-N+1) \tag{36}$$

where $N = 2^m$ and the coefficients $a_0, a_1, \cdots, a_N$ are to be found, we observe that it is tantamount to writing the transfer function $\bar{\bar{H}}$ in (34) as

$$\bar{\bar{H}} = a_0\bar{\bar{I}} + a_1\bar{\bar{A}} + \cdots + a_{N-1}\bar{\bar{A}}^{N-1} \tag{37}$$

since the Walsh transformation of (36) should lead to (33). Pre- and postmultiplying (37) by $\bar{\bar{W}}$ and $\bar{\bar{W}}^{-1}$, respectively, we obtain, by virtue of (11),

$$\bar{\bar{W}}\bar{\bar{H}}\bar{\bar{W}}^{-1} = a_0\bar{\bar{I}} + a_1\bar{\bar{I}}_{-1} + \cdots + a_{N-1}\bar{\bar{I}}_{-(N-1)}. \tag{38}$$

The coefficients $a_0, a_1, \cdots, a_N$ can be determined easily from (38) for any given $\bar{\bar{H}}$ by inspection.

For the present example, $\bar{\bar{A}}$ has been given in (17) for $N = 8$. From (34), we have

$$\bar{\bar{H}} = \begin{bmatrix} 0.111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.029 & -0.029 & 0 & 0 & -0.147 & -0.147 & 0 \\ 0 & 0.029 & 0.029 & 0 & 0 & -0.147 & 0.147 & 0 \\ 0 & 0 & 0 & 0 & -0.200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.200 & 0 & 0 & 0 & 0 \\ 0 & -0.147 & 0.147 & 0 & 0 & -0.265 & -0.265 & 0 \\ 0 & 0.147 & 0.147 & 0 & 0 & 0.265 & -0.265 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{39}$$

Pre- and postmultiplying $\bar{H}$ in (39) by $\bar{W}$ and $\bar{W}^{-1}$, respectively,and observing (38), we obtain

$$\bar{W}\bar{H}\bar{W}^{-1} = \begin{bmatrix} -0.170 & 0.336 & -0.170 & 0.089 & -0.052 & 0.042 & -0.052 & 0.089 \\ 0.089 & -0.170 & 0.336 & -0.170 & 0.089 & -0.052 & 0.042 & -0.052 \\ -0.052 & 0.089 & -0.170 & 0.336 & -0.170 & 0.089 & -0.052 & 0.042 \\ 0.042 & -0.052 & 0.089 & -0.170 & 0.336 & -0.170 & 0.089 & -0.052 \\ -0.052 & 0.042 & -0.052 & 0.089 & -0.170 & 0.336 & -0.170 & 0.089 \\ 0.089 & -0.052 & 0.042 & -0.052 & 0.089 & -0.170 & 0.336 & -0.170 \\ -0.170 & 0.089 & -0.052 & 0.042 & -0.052 & 0.089 & -0.170 & 0.336 \\ 0.336 & -0.170 & 0.089 & -0.052 & 0.042 & -0.052 & 0.089 & -0.170 \end{bmatrix}$$

$$= -0.170I + 0.089I_{-1} - 0.052I_{-2} + 0.042I_{-3} - 0.052I_{-4} + 0.089I_{-5} - 0.170I_{-6} + 0.336I_{-7} \quad (40)$$

which is in the form of (38). We can then write $\bar{H}$ in (34) in the form of (37)

$$\bar{H} = -0.170\bar{I} + 0.089\bar{\bar{A}} - 0.052\bar{\bar{A}}^2 + 0.042\bar{\bar{A}}^3 - 0.052\bar{\bar{A}}^4 + 0.089\bar{\bar{A}}^5 - 0.170\bar{\bar{A}}^6 + 0.336\bar{\bar{A}}^7. \quad (41)$$

Since $\bar{Y} = \bar{H}\bar{X}$, as defined in (33), we have the following recursive formula for the given system:

$$\bar{y} = -0.170\bar{x} + 0.089\bar{x}_{-1} - 0.052\bar{x}_{-2} + 0.042\bar{x}_{-3} - 0.052\bar{x}_{-4} + 0.089\bar{x}_{-5} - 0.170\bar{x}_{-6} + 0.336\bar{x}_{-7}. \quad (42)$$

Equation (42) expresses the output $\bar{y}$ as a linear combination of the input $\bar{x}$ shifted by various amounts. It would be easy to devise a hardware implementation of (42) as the solution of the difference equation given in (31).

Conclusion

Time-shift theorems for Walsh transforms of functions subject to nondyadic as well as dyadic time displacements have been presented. Spectrum-conversion matrices have been defined and a relation between a function with an ordinary shift and that with a dyadic shift has been established. Methods for solving difference equations by Walsh transformation have been shown.

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An Algorithm for Sequency Ordering of Hadamard Functions

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***Abstract*—A simple algorithm is developed for obtaining the sequency vector of high-order Hadamard transform matrices without the need for converting the order of individual Hadamard functions to sequency.**

***Index Terms*—Hadamard functions, sequency (ordering), Walsh functions.**

Hadamard functions [1], like Walsh functions [2], are binary-valued functions which form a complete orthonormal set on [0,1). However, unlike Walsh functions, the order h of a Hadamard function $H(h,t)$ is not equal to its sequency, which is defined as the number of zero crossings in the unit interval. It is known that the sequency of a Hadamard function can be determined by finding the inverse Gray code of the bit-reverse of the binary representation of its order [3], [4]. Concisely, if the binary form of h is

$$(h)_2 = (b_{n-1}b_{n-2}\cdots b_2b_1), \tag{1}$$

then its bit-reverse is

$$(h)_2' = (b_1b_2\cdots b_{n-2}b_{n-1}). \tag{2}$$

The inverse Gray code of $(h)_2'$ in (2) is

$$(w)_2 = (c_1c_2\cdots c_{n-2}c_{n-1}) \tag{3}$$

with

$$c_i = \sum_{j=1}^{i} b_j \tag{4}$$

where Σ denotes a dyadic or logical summation. $(w)_2$ is the binary form of the sequency of $H(h,t)$.

The Hadamard matrix $\overline{\overline{H}}_n$ transforms a set of $N = 2^n$ sampled data in the time domain to a discrete set of N values in the sequency domain. The hth row of $\overline{\overline{H}}_n$ represents the values of the Hadamard function $H(h,t)$ where h varies from 0 to $N - 1$. The use of Hadamard transformation is convenient because of the ease of generation of a Hadamard matrix of an arbitrary order from the basic Hadamard matrix

$$\overline{\overline{H}}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{5}$$

by the following recursive formula:

$$\overline{\overline{H}}_{n_1+n_2} = \overline{\overline{H}}_{n_1} \otimes \overline{\overline{H}}_{n_2} \tag{6}$$

where $\otimes$ denotes a Kronecker product [3]–[5]. In particular, for $n = 3$, the 8×8 Hadamard matrix and the sequency ordering of its rows are, respectively,

$$\begin{array}{cc} \text{Hadamard Transform Matrix} & \text{Sequency Vector} \\ \overline{\overline{H}}_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} & \overline{S}_3 = \begin{bmatrix} 0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5 \end{bmatrix} \end{array} \tag{7}$$

where the sequency ordering is expressed as a column vector for convenience. If the rows of the Hadamard matrix are rearranged in increasing sequency, it becomes a Walsh matrix which finds important applications in image processing and other areas. The question arises: what are the sequency vectors of the 16×16 Hadamard matrix and of 32×32 and higher order Hadamard matrices? The answer to this question is not obvious and the bit-reverse inverse Gray code procedure applied to each row of the Hadamard matrix is tedious. In the following we develop an algorithm for obtaining the entire sequency vector by inspection.

Assertion: Let $\overline{S}_{n-1}$ be the sequency vector for the $2^{n-1} \times 2^{n-1}$ Hadamard matrix:

$$\overline{S}_{n-1} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2^{n-1}-1} \end{bmatrix} \tag{8}$$

Then the sequency vector $\overline{S}_n$ for the $2^n \times 2^n$ Hadamard matrix is

$$\overline{S}_n = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{2^n-1} \end{bmatrix} \tag{9}$$

where

$$d_{2k} = c_k \tag{10}$$

$$d_{2k+1} = (2^n - 1) - c_k, \qquad k = 0, 1, \cdots, 2^{n-1} - 1. \tag{11}$$

This assertion can be expressed more concisely as $S_n(2k) = S_{n-1}(k)$ and $S_n(2k + 1) = (2^n - 1) - S_{n-1}(k)$.

Proof: We shall prove the two recursive formulas (10) and (11) separately.

i) $d_{2k} = c_k$:

Let the binary representation of the sequency order k of $\overline{S}_{n-1}$ be

$$(k)_2 = (a_{n-1}a_{n-2}\cdots a_2a_1); \tag{12}$$

then its bit-reverse $(k)_2'$ is

$$(k)_2' = (a_1a_2\cdots a_{n-2}a_{n-1}). \tag{13}$$

From (14) the binary representation of sequency order $2k$ in $\overline{S}_n$ is seen to be

$$(2k)_2 = (a_{n-1}a_{n-2}\cdots a_2a_10) \tag{14}$$

whose bit-reverse is

$$(2k)_2' = (0a_1a_2\cdots a_{n-2}a_{n-1}). \tag{15}$$

It is evident that the inverse Gray code of $(2k)_2'$ in (15) is numerically the same as that of $(k)_2'$ in (13). Hence, $d_{2k} = c_k$.

ii) $d_{2k+1} = (2^n - 1) - c_k$:

Because of (6), the above relation is equivalent to

$$d_{2k} + d_{2k+1} = 2^n - 1. \tag{16}$$

From (10), we can write immediately

$$(2k + 1)_2 = (a_{n-1}a_{n-2}\cdots a_2a_11) \tag{17}$$

and

$$(2k + 1)_2' = (1a_1a_2\cdots a_{n-2}a_{n-1}). \tag{18}$$

Let $(0l_1l_2\cdots l_i\cdots l_{n-1})$ and $(1\bar{l}_1\bar{l}_2\cdots \bar{l}_i\cdots \bar{l}_{n-1})$ be, respecti[illegible] inverse Gray code of $(2k)_2'$ in (15) and $(2k + 1)_2'$ in (18). We have, fr[illegible] (4),

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$$l_i = 0 \oplus a_1 \oplus a_2 \oplus \cdots \oplus a_i, \qquad i = 1,2,\cdots,n-1 \tag{19}$$

and

$$\bar{l}_i = 1 \oplus a_1 \oplus a_2 \oplus \cdots \oplus a_i, \qquad i = 1,2,\cdots,n-1 \tag{20}$$

where $\oplus$ denotes dyadic or logical addition. It is obvious from (19) and (20) that

$$l_i + \bar{l}_i = 1. \tag{21}$$

Hence,

$$\begin{aligned}(0 l_1 l_2 \cdots l_{n-1}) &+ (1 \bar{l}_1 \bar{l}_2 \cdots \bar{l}_{n-1}) \\ &= (1\,1\,1 \cdots 1) = 2^n - 1\end{aligned} \tag{22}$$

and (16) follows directly. The proof of our assertion is now complete.

An algorithm based on recursive formulas (10) and (11) can be devised to generate $\bar{S}_n$ and $\bar{S}_{n-1}$. In order to find $\bar{S}_4$ and $\bar{S}_5$ knowing $\bar{S}_3$ in (7), we note that, for the 16×16 Hadamard matrix, $n = 4$ and $2^n - 1 = 15$. The transpose of $\bar{S}_4$ can be written down immediately by inspection:

$$\bar{S}_4^t = [0\ 15\ 7\ 8\ 3\ 12\ 4\ 11\ 1\ 14\ 6\ 9\ 2\ 13\ 5\ 10]. \tag{23}$$

Similarly, for the 32×32 Hadamard matrix, $n = 5, 2^n - 1 = 31$, and the transpose of the sequency vector $\bar{S}_5$ is

$$\begin{aligned}\bar{S}_5^t = [&0\ 31\ 15\ 16\ 7\ 24\ 8\ 23\ 3\ 28\ 12\ 19\ 4\ 27\ 11\ 20 \\ &1\ 30\ 14\ 17\ 6\ 25\ 9\ 22\ 2\ 29\ 13\ 18\ 5\ 26\ 10\ 21].\end{aligned} \tag{24}$$

Each number in (23) and (24) represents the sequency of the corresponding Hadamard function in the Hadamard matrix.

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A Generalized Orthogonal Transformation Matrix

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Abstract—**A procedure is described for generating a two-parameter orthogonal transformation matrix which reduces to the Fourier and Hadamard transformation matrices under special conditions. This generalized transformation matrix is particularly useful for multidimensional signal processing on a real-time basis because it preserves a proper relationship in the transform domain.**

Index Terms—**Fourier transform, Hadamard transform, orthogonal transform, raster scan, signal processing.**

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Introduction

A procedure frequently used in two-dimensional image processing is to scan the image sequentially in time (raster scan), converting it into a one-dimensional digital signal and then processing it in blocks in the transform domain [1], [2]. The scanning process, in effect, transmits a two-dimensional matrix in the form of a one-dimensional column matrix (vector). For example, the matrix signal or image

$$\bar{x} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}. \tag{1}$$

would be scanned and transmitted as $\bar{x}$ whose transposition, $\bar{x}^t$, is

$$\bar{x}^t = [a_1 \quad a_2 \quad a_3 \quad b_1 \quad b_2 \quad b_3 \quad c_1 \quad c_2 \quad c_3]. \tag{2}$$

However, this conversion alters the relative positions of the elements, and the autocorrelation of $\bar{x}$ [or of $\bar{x}^t$ in (2)] is obviously not the same as that of $\bar{x}$ in (1). It is seen that, in $\bar{x}$, b_1 is close to a_1 and c_1 whereas, in $\bar{x}$ or $\bar{x}^t$, b_1 is quite far from a_1 and c_1 but is now close to a_3. Hence, the scanning process changes the character of the original image, and processing in the transform domain after scanning may not yield the desired results.

This correspondence introduces a two-parameter generalized transformation matrix which reduces to the Fourier and Hadamard transformation matrices under special conditions. It is structured on the m-ary number system using m-adic (modulo m) operations. The generalized matrix can be generated from a basic core matrix in a relatively simple manner. It will be shown that the new transformation matrix with appropriate parameters will preserve the proper relationship in the transform domain before and after scanning and that m-adic operations based on the m-ary number system are inherently relevant in the processing of $m \times m$ matrix signals. Ahmed *et al.* [3] discussed the generation of a family of discrete orthogonal transforms for a given periodic data sequence. Their emphasis was in factoring these transform matrices into a product of sparse matrices and was not concerned with preserving the character of the scanned image.

Preliminaries

Consider a simple 4×4 matrix signal

$$\bar{x} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 2 & 3 & 0 \\ 2 & 4 & 1 & 1 \\ 3 & 1 & 0 & 3 \end{bmatrix}. \tag{3}$$

Its two-dimensional discrete Fourier transform, $\bar{X}_f$, is easily obtained [4], [5]. We have

$$\bar{X}_f = \bar{F}_4^{-1}\bar{x}\bar{F}_4 = \begin{bmatrix} 31 & 1 & -1 & 1 \\ 2+j & -6+j11 & 2-j3 & 2-j \\ 5 & -1 & -3 & -1 \\ 2-j & 2+j & 2+j3 & -6-j11 \end{bmatrix}. \tag{4}$$

where the 4×4 Fourier transformation matrix $\bar{F}_4$ is

$$\bar{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}. \tag{5}$$

When raster scan is used to convert $\bar{x}$ in (3) to a one-dimensional 16×1 column vector $\bar{x}$, a 16×16 transformation matrix $\bar{F}_{16}$ is conventionally used to obtain its Fourier transform $\bar{X}_f$. To save space, we write

$$\begin{aligned} \bar{X}_f^t &= \{\bar{F}_{16}\bar{x}\}^t = \bar{x}^t\bar{F}_{16}^t \\ &= [31, \quad 0.058 - j2.935, \quad -j3.828, \quad -1.141 + j1.539, \\ &\quad 1, \quad 2.555 + j12.61, \quad -j1.828, \quad -1.473 + j0.136, \\ &\quad -1, \quad -1.473 - j0.136, \quad j1.828, \quad 2.255 - j12.61, \\ &\quad 1, \quad -1.141 - j1.539, \quad j3.828, \quad 0.058 + j2.935], \end{aligned} \tag{6}$$

where

$$\bar{F}_{16} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\pi/8} & e^{j\pi/4} & \cdots & e^{j15\pi/8} \\ 1 & e^{j\pi/4} & e^{j\pi/2} & \cdots & e^{j15\pi/4} \\ \vdots & \vdots & & & \\ 1 & e^{j15\pi/8} & e^{j15\pi/4} & \cdots & e^{j225\pi/4} \end{bmatrix} \tag{7}$$

and

$$\bar{x}^t = [2 \quad 1 \quad 3 \quad 4 \quad 1 \quad 2 \quad 3 \quad 0 \quad 2 \quad 4 \quad 1 \quad 1 \quad 3 \quad 1 \quad 0 \quad 3]. \tag{8}$$

Obviously, $\bar{X}_f^t$ in (6), when rearranged in a two-dimensional form, is not the same as $\bar{X}_f$ in (4). We seek a new orthogonal transform matrix which will alleviate this discrepancy.

The Core Matrix

In general, a two-dimensional image signal $\bar{x}$ may comprise $m \times m$ elements; that is, there may be m rows each containing m elements. Raster scanning makes the first element of the second row (which is spatially close to the first element of the first row) the $(m + 1)$th element. Conversely, the reconstruction of an image from a one-dimensional digital signal puts the $(m + 1)$th element at the head of the second row which is close to the first element of the first row. This suggests a similarity with the structure of an m-ary number system.

We define an $m \times m$ basic orthogonal transformation matrix as follows:

$$\bar{G}_{m,1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/m} & e^{-j4\pi/m} & \cdots & e^{-j(m-1)2\pi/m} \\ 1 & e^{-j4\pi/m} & e^{-j8\pi/m} & & e^{-j(m-1)4\pi/m} \\ \vdots & & & & \\ 1 & e^{-j(m-1)2\pi/m} & \cdots & \cdots & e^{-j(m-1)(m-1)2\pi/m} \end{bmatrix}. \tag{9}$$

$\bar{G}_{m,1}$ is an m-ary matrix of the first order and it is easy to verify that (on a matrix denotes the transpose complex conjugate, or adjoint, of the matrix)

$$\bar{G}_{m,1}\bar{G}_{m,1}^{*} = m\bar{I}, \tag{10}$$

where $\bar{I}$ is an $m \times m$ identity matrix. We shall designate the orthogonal $\bar{G}_{m,1}$ as the *core matrix*. We note here that m is *not* restricted to be an integral power of 2.

The Generalized Transformation Matrix

The core matrix $\bar{G}_{m,1}$ may be extended to higher orders by Kronecker multiplication [4], [6]:

$$\bar{G}_{m,n} = \bar{G}_{m,n-1} \otimes \bar{G}_{m,1}. \tag{11}$$

where $\bar{G}_{m,n}$ is an m-ary matrix of the nth order. It has a dimension $m^n \times m^n$ and is also orthogonal. We note that $\bar{G}_{m,1}(n = 1)$ is a Fourier matrix and that $\bar{G}_{2,n}(m = 2)$ is a Hadamard matrix. Hence, $\bar{G}_{m,n}$ may be called a *generalized transformation matrix* which reduces to Fourier and Hadamard matrices for particular values of the parameters m and n. It is important to observe that, in general,

$$\bar{G}_{m^2,1} = \bar{F}_{m^2} \neq \bar{G}_{m,2} \tag{12}$$

and

$$\bar{G}_{2^k,1} \neq \bar{G}_{2,k} = \bar{H}_{2^k}, \tag{13}$$

where $\bar{H}_{2^k}$ is the Hadamard matrix of the 2^kth order [6].

For the 4×4 matrix signal in (3), we transform it by $\bar{G}_{4,1}$. Since $\bar{G}_{4,1}$ is the same as $\bar{F}_4$ in (5), we obtain the same result in the transform domain; that is, $\bar{X}_g = \bar{G}_{4,1}^{-1}\bar{x}\bar{G}_{4,1}$, which equals the $\bar{X}_f$ in (4). With raster scan, the resulting $\bar{x}$ should be transformed by the 16×16 matrix $\bar{G}_{4,2}$:

$$\bar{G}_{4,2} = \bar{G}_{4,1} \otimes \bar{G}_{4,1} = \begin{bmatrix} \bar{G}_{4,1} & \bar{G}_{4,1} & \bar{G}_{4,1} & \bar{G}_{4,1} \\ \bar{G}_{4,1} & -j\bar{G}_{4,1} & -\bar{G}_{4,1} & j\bar{G}_{4,1} \\ \bar{G}_{4,1} & -\bar{G}_{4,1} & \bar{G}_{4,1} & -\bar{G}_{4,1} \\ \bar{G}_{4,1} & j\bar{G}_{4,1} & -\bar{G}_{4,1} & -j\bar{G}_{4,1} \end{bmatrix}. \tag{14}$$

It is readily shown that

$$\bar{X}_g = \bar{G}_{4,2}\bar{x} = [31,\ 1\ \ -1,\ 1,\ 2+j,\ \ -6+j11,\ 2-j3,\ 2-j,\ 5,-1,\ -3,\ -1,\ 2-j,\ 2+j,\ 2+j3,\ -6-j11], \tag{15}$$

which is the same as the one-dimensional scanned version of $\bar{X}_g$ or of $\bar{X}_f$ in (4). The proper relationship in the transform domain for the two-dimensional signal is therefore preserved for the one-dimensional signal after scanning.

Correlation Functions

We have seen from (11) that a generalized transformation matrix of a high order can be generated from the core matrix by Kronecker multiplication. The core matrix is basically built on an m-ary number system. The propriety of using m-adic arithmetic in dealing with $m \times m$ two-dimensional images can be further demonstrated by considering the correlation functions.

The (i_1, i_2)th element of the cross-correlation matrix of two $m \times m$ signals $\bar{x}$ and $\bar{y}$ can be written as

$$R_{xy}(i_1, i_2) = \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} x(i_1 \oplus j_1, i_2 \oplus j_2) \cdot y^*(j_1, j_2); \quad \begin{matrix} 0 \le i_1, i_2 \le m-1 \\ 0 \le j_1, j_2 \le m-1 \end{matrix} \tag{16}$$

where $*$ denotes complex conjugate. The symbol $\oplus$ represents an m-adic (or modulo m with no carry) addition. Raster scanning converts the two-dimensional $m \times m$ matrices $\bar{x}$ and $\bar{y}$ into one-dimensional $m^2 \times 1$ vectors $\bar{x}$ and $\bar{y}$. For example, the (i_1, i_2)th element in $\bar{x}$ will appear as the ith element in $\bar{x}$, where

$$i = i_1 m + i_2, \qquad 0 \le i_1, i_2 \le m-1. \tag{17}$$

$R_{xy}(i_1, i_2)$ in (16) therefore can be rewritten as

$$R_{xy}(i) = \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} x\{(i_1 \oplus j_1)m + (i_2 \oplus j_2)\}\, y^*(j_1 m + j_2) = \sum_{j=0}^{m^2-1} x(i \oplus j)\, y^*(j), \tag{18}$$

which is exactly the element in the one-dimensional $\bar{R}_{xy}$ corresponding to the (i_1, i_2)th element in the two-dimensional $\bar{R}_{xy}$ shown in (16).

The second-order generalized transformation matrix $\bar{G}_{m,2}$ must be used on $\bar{R}_{xy}$ to obtain the $m^2 \times 1$ power-density vector $\bar{S}_{xy}$:

$$\bar{S}_{xy} = \bar{G}_{m,2}\bar{R}_{xy} = \bar{X}_g \cdot \bar{Y}_g^*, \tag{19}$$

where $\bar{X}_g \cdot \bar{Y}_g^*$ is a termwise multiplication [7]. $\bar{S}$ in (19) rearranged as $m \times m$ matrix will be the same as

$$\bar{S}_{xy} = \bar{G}_{m,1}^{-1}\bar{R}_{xy}\bar{G}_{m,1}. \tag{20}$$

The same conclusion cannot be drawn if $\bar{G}_{m,2}$ and $\bar{G}_{m,1}$ are replaced by $\bar{F}_{m^2}$ and $\bar{F}_m$, respectively. As a consequence, a distorted power-density spectrum would be obtained if the conventional Fourier or Hadamard transformation matrix is used with a scanned signal.

For the given 4×4 matrix signal $\bar{x}$ in (3), the autocorrelation matrix $\bar{R}_x$ is, from (16),

$$\bar{R}_x = \begin{bmatrix} 85 & 60 & 44 & 60 \\ 58 & 40 & 58 & 78 \\ 40 & 62 & 80 & 62 \\ 58 & 78 & 58 & 40 \end{bmatrix} \tag{21}$$

and the power-density matrix is, from (20),

$$\bar{S}_x = \begin{bmatrix} 961 & 1 & 1 & 1 \\ 5 & 157 & 13 & 5 \\ 25 & 1 & 9 & 1 \\ 5 & 5 & 13 & 157 \end{bmatrix}. \tag{22}$$

Inspection of (15) and (22) reveals immediately that the relation in (19) is satisfied.

If quadri-adic addition is not used, the transposed autocorrelation vector of the scanned $\bar{x}$ would be

$$\bar{R}_x'' = [85, 54, 46, 78, 58, 40, 74, 68, 40, 68, 74, 40, 58, 78, 46, 54], \tag{23}$$

which bears no simple relation with $\bar{R}_x$ in (21). Furthermore, the discrete Fourier transform of $\bar{R}_x'$ in (23), after rearrangement, is

$$\bar{S}_x' = \begin{bmatrix} 961 & 8.617 & 14.66 & 3.668 \\ 1 & 165.5 & 3.343 & 2.187 \\ 1 & 2.187 & 3.343 & 165.5 \\ 1 & 3.668 & 14.66 & 8.617 \end{bmatrix}. \tag{24}$$

Compared with $\bar{S}_x$ in (22), $\bar{S}_x'$ in (24) is obviously incorrect; so also will be the results of subsequent steps of signal processing. On the other hand, when the generalized matrix $\bar{G}_{4,2}$, derived in accordance with (14), is used to transform the scanned version, $\bar{R}_x$, of $\bar{R}_x$ in (21), we obtain

$$\bar{S}_x = \bar{G}_{4,2}\bar{R}_x, \tag{25}$$

which, when rearranged in a two-dimensional form, restores the power-density matrix $\bar{S}_x$ in (22).

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Noise-Error Determination of Combinational Circuits by Walsh Functions

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Abstract—The stochastic behavior of digital combinational circuits is analyzed by the use of Walsh functions. An n-input Boolean function is represented as a Walsh series and the error caused by noise is measured in terms of a *distance* which is the fraction of the time that the system output due to noise-corrupted signal differs from that due to signal alone. It is shown that the error can be expressed as the sum of two parts: one part depends only on noise statistics, and the other on both signal and noise. Some interesting properties of both parts are discussed and typical examples are given.

Key Words: Noise error, digital combinational circuits, Walsh functions.

I. INTRODUCTION

THE OUTPUT behavior of linear (and some nonlinear) analog systems in response to stochastic inputs can be determined in a large number of cases. However, no such comparable facility exists for digital systems, such as those employing threshold-type devices and discrete-valued waveforms. Given input statistics and digital-system transformations, it is, in general, a formidable task to obtain the output statistics. An obvious reason for this difficulty is that digital-device (such as a transistor-transistor logic (TTL) gate) models are not linear RLC models and cannot be handled by the usual linear algebra and calculus. Logical algebra and an associated calculus that is digital in nature are required [1]. In this connection, Walsh functions have been found to be relevant [2]-[5]. This paper expresses an n-input Boolean function as a Walsh series and defines the error at the output of a combinational circuit caused by noise as the *distance* between the responses to the signal and to the noise-corrupted signal inputs. The use of Walsh functions appears natural here and does, in fact, facilitate the computation of the system error [6], [7].

It will be shown that the error can be obtained as the sum of two parts: one part depends on noise statistics only, and the other depends on the characteristics of both signal and noise. For independent and identically distributed noise processes,

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the first part is invariant over each of certain equivalence classes of Boolean functions. Under certain conditions of noise and signal-component processes, the system error is expressible as a polynomial function of the expected values of signal and of noise. Some interesting properties of the error polynomial will be discussed. The Boolean functions of two combinational circuits are studied, and the system errors are computed and plotted for various statistics of signal and noise.

II. PRELIMINARY CONSIDERATIONS

Any n-input combinational circuit S can be represented by a Boolean function that maps every binary n-tuple to either 0 or 1. Let the set of all binary n-tuples be denoted by V_n. Then

$$V_n = \{\langle x_{n-1}, x_{n-2}, \cdots, x_0 \rangle : x_i \in \{0, 1\}, \text{ for } 0 \leqslant i < n\}. \tag{1}$$

Since an n-tuple can be considered as a vector, it is convenient to denote $\langle x_{n-1}, x_{n-2}, \cdots, x_0 \rangle$ by $\bar{x}$. Let x stand for the integer whose binary representation is $\bar{x}$; we have

$$x = \sum_{j=0}^{n-1} x_j 2^j. \tag{2}$$

Equation (2) establishes an equivalence between $\{0, 1, 2, \cdots, 2^n - 1\}$, denoted by D_n, and V_n in (1). Any function or operation defined on V_n is thus also defined on D_n. Fig. 1 shows a schematic diagram of an n-input combinational circuit with a single binary output.

Dyadic operations are fundamental to Walsh functions and their applications. Dyadic addition, denoted by $\oplus$ (on V_n, and hence on D_n), is defined as follows. For any $\bar{x}$ and $\bar{y}$ in V_n,

$$\bar{x} \oplus \bar{y} = \langle x_{n-1} \oplus y_{n-1}, x_{n-2} \oplus y_{n-2}, \cdots, x_0 \oplus y_0 \rangle \in V_n \tag{3}$$

where

$$0 \oplus 0 = 1 \oplus 1 = 0 \tag{3a}$$

and

$$0 \oplus 1 = 1 \oplus 0 = 1. \tag{3b}$$

In fact, $(V_n, \oplus)$ forms the dyadic group for which Walsh functions are the character functions [8].

For an input $\bar{X}(t) = \{\langle X_{n-1}(t), X_{n-2}(t), \cdots, X_0(t)\rangle$: for $t \in T$, $X_i(t) \in \{0, 1\}$, for $0 \leqslant i < n\}$, where each $X_i(t)$ is a binary stochastic process, the output of a given combinational circuit S is also a binary stochastic process $\{S(\bar{X}(t)) \in \{0, 1\}$: $t \in T\}$. We define a *distance* $D_S[\bar{X}(t), \bar{Y}(t)]$ between the responses to two inputs $\bar{X}(t)$ and $\bar{Y}(t)$ as the fraction of the time that the output $S(\bar{X}(t))$ differs from $S(\bar{Y}(t))$. If $\bar{X}(t)$ is the signal input process in the absence of noise and $\bar{X}_N(t) = \{\langle X_{N,n-1}(t), X_{N,n-2}(t), \cdots, X_{N,0}(t)\rangle \in V_n$: $t \in T\}$ a noise-corrupted input process, then, in general, $S(\bar{X}(t)) \neq S(\bar{X}_N(t))$ and we write the error at the output due to noise at the input

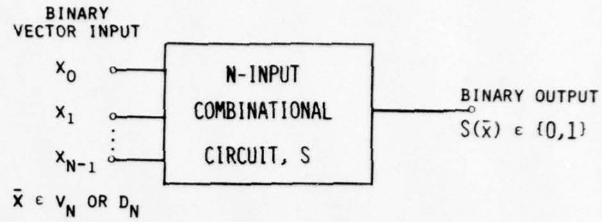


Fig. 1. Schematic diagram of an n-input combinational circuit.

as

$$\xi_S = D_S[\bar{X}(t), \bar{X}_N(t)]. \tag{4}$$

A more precise definition for ξ_S will be given later.

We assume that the noise is also a binary process $\bar{N}(t) = \{\langle N_{n-1}(t), N_{n-2}(t), \cdots, N_0(t)\rangle \in V_n$: $t \in T\}$ and is added to $\bar{X}(t)$ dyadically to yield

$$\bar{X}_N(t) = \bar{X}(t) \oplus \bar{N}(t). \tag{5}$$

The dyadic addition in (5) is appropriate inasmuch as a combined input digit is in error if, and only if, the corresponding noise digit is 1. Consequently, Walsh functions can be used to advantage in noise-error determination.

Walsh functions, denoted by Wal $(\cdot)$, form an orthogonal basis on V_n [8], [9]. For $i \in D_n$ and $\bar{x} \in V_n$, the Hadamard-ordered Walsh functions [10], [11], are

$$\mathrm{Wal}\,(i, \bar{x}) = (-1)^{\sum_{j=0}^{n-1} i_j x_j} \tag{6}$$

where the notation conforms with that in (2). Obviously, Wal $(i, \bar{x}) \in \{1, -1\}$. The following properties can be readily verified [8], [12]:

a) $\mathrm{Wal}\,(i, x) = \mathrm{Wal}\,(x, i)$ (7)

b) $\displaystyle\sum_{\bar{x} \in V_n} \mathrm{Wal}\,(i, \bar{x})\,\mathrm{Wal}\,(j, \bar{x}) = 2^n \delta_{ij}$ (8)

where δ_{ij} is the Kronecker delta

c) $\mathrm{Wal}\,(i, \bar{x} \oplus \bar{y}) = \mathrm{Wal}\,(i, \bar{x})\,\mathrm{Wal}\,(i, \bar{y})$. (9)

III. EQUIVALENT BOOLEAN FUNCTIONS AND THEIR WALSH REPRESENTATION

Because Walsh functions form an orthogonal basis on V_n, we can represent any Boolean function as a Walsh series. We write, for any n-input Boolean function S,

$$S(\bar{x}) = \sum_{i \in D_n} b_i \,\mathrm{Wal}\,(i, \bar{x}) \tag{10}$$

where

$$b_i = 2^{-n} \sum_{\bar{x} \in V_n} S(\bar{x})\,\mathrm{Wal}\,(i, \bar{x}). \tag{11}$$

Two Boolean functions $S_1(\bar{x})$ and $S_2(\bar{x})$ on V_n are said to be equivalent [13] if there is a sequence of permutations and complementations of some of the variables $\langle x_{n-1}, x_{n-2}, \cdots, x_0 \rangle$ to produce $\langle y_{n-1}, y_{n-2}, \cdots, y_0 \rangle$ such that, for every $\bar{x} \in V_n$,

$$S_1(\bar{y}) \oplus S_2(\bar{x}) = d, \qquad \text{for } d \in \{0, 1\}. \tag{12}$$

A. Assertion I (See Appendix I for Proof)

Let an n-input Boolean function $S(\bar{x})$ as represented in (10) be transformed under an equivalence operation (permutations or complementations) to

$$Q(\bar{x}) = \sum_{i \in D_n} q_i \operatorname{Wal}(i, \bar{x}). \tag{13}$$

a) If $Q(\bar{x})$ is the complement of $S(\bar{x})$, then

$$q_i = \begin{cases} 1 - b_0, & \text{for } i = 0 \\ -b_i, & \text{for } i \in D_n - \{0\} = \{1, 2, \cdots, 2^n - 1\}. \end{cases} \tag{14}$$

b) If out of $\bar{x} \in V_n$, the inputs $x_{j_1}, x_{j_2}, \cdots, x_{j_m}$, where $m \leqslant n$, are complemented, then

$$q_i = b_i \operatorname{Wal}(i, h) \tag{15}$$

where

$$h = \sum_{k=1}^{m} 2^{j_k}. \tag{15a}$$

c) If x_α and x_β $(0 \leqslant \alpha, \beta < n)$ are interchanged, then

$$q_i = \begin{cases} b_i, & \text{for those } i \text{ where } i_\alpha = i_\beta \\ b_{i \oplus 2^\alpha \oplus 2^\beta}, & \text{for those } i \text{ where } i_\alpha \neq i_\beta. \end{cases} \tag{16}$$

This assertion will be used to prove that the part of the error which depends only on noise statistics is invariant over complementations in Boolean functions and over an interchange of inputs under certain conditions.

IV. NOISE ERROR IN COMBINATIONAL CIRCUIT

We have defined previously a distance $D_S[\bar{X}(t), \bar{Y}(t)]$ between the responses of a combinational circuit S to the inputs $\bar{X}(t)$ and $\bar{Y}(t)$ in a qualitative manner. For stochastic inputs over a discrete time domain T,

$$D_S[\bar{X}(t), \bar{Y}(t)] = E\left\{\frac{1}{|T|} \sum_{t \in T} [S(\bar{X}(t)) \oplus S(\bar{Y}(t))]\right\} \tag{17}$$

where $D_S[\cdot, \cdot]$ satisfies the following properties:

a) $D_S[\bar{X}(t), \bar{Y}(t)] = D_S[\bar{Y}(t), \bar{X}(t)] \geqslant 0$ (18)

b) $D_S[\bar{X}(t), \bar{X}(t)] = 0$ (19)

c) $D_S[\bar{X}(t), \bar{Y}(t)] \leqslant D_S[\bar{X}(t), \bar{Z}(t)] + D_S[\bar{Z}(t), \bar{Y}(t)].$ (20)

Combining (4), (1[illegible] and (10), we have the following Walsh representation for the error ξ_S at the output of the combinational circuit S for signal $\bar{X}(t)$ corrupted by noise $\bar{N}(t)$.

$$\xi_S = E\left\{\frac{1}{|T|} \sum_{t \in T} \left[\sum_{i \in D_n} b_i \operatorname{Wal}(i, \bar{X}(t)) - \sum_{i \in D_n} b_i \operatorname{Wal}(i, \bar{X}_N(t))\right]^2\right\} \tag{21}$$

where the identity $a \oplus b = (a - b)^2$ for $a, b \in \{0, 1\}$ has been used. The following relations hold:

$$E\{[1 - \operatorname{Wal}(i, \bar{N}(t))]^2\} = 2\{1 - E[\operatorname{Wal}(i, \bar{N}(t))]\} \tag{22}$$

which vanishes if $i = 0$; and

$$E\{[1 - \operatorname{Wal}(i, \bar{N}(t))][1 - \operatorname{Wal}(j, \bar{N}(t))]\} = 0 \tag{23}$$

if i or $j = 0$. By using (5), (22), and (23), we can put (21) in the following form:

$$\xi_S = \xi_S(N) + \xi_S(N, X) \tag{24}$$

where

$$\xi_S(N) = \frac{2}{|T|} \sum_{t \in T} \sum_{i \in D_n} b_i^2 [1 - E\{\operatorname{Wal}(i, \bar{N}(t))\}] \tag{25}$$

which depends only on noise, and

$$\xi_S(N, X) = \frac{1}{|T|} \sum_{t \in T} \sum_{i \neq j : i, j \in D_n} b_i b_j \, E\{[1 - \operatorname{Wal}(i, \bar{N}(t))] \cdot [1 - \operatorname{Wal}(j, \bar{N}(t))] \cdot \operatorname{Wal}(i \oplus j, X(t))\} \tag{26}$$

which depends on both signal and noise. Note that D_n may be replaced by $D_n - \{0\}$. Equations (24)–(26) are general results for system errors.

Under the complementation operations in Assertion I: a), b), which transform S to Q, equations (14) and (15) indicate that $q_i^2 = b_i^2$ for $i \neq 0$. Hence, $\xi_Q(N) = \xi_S(N)$. Furthermore, using (16) and (25), we have

$$\xi_Q(N) = \frac{2}{|T|} \sum_{t \in T} \sum_{i_\alpha = i_\beta} b_i^2 [1 - E\{\operatorname{Wal}(i, \bar{N}(t))\}] + \frac{2}{|T|} \sum_{t \in T} \sum_{i_\alpha \neq i_\beta} b_i^2 [1 - E\{\operatorname{Wal}(i \oplus 2^\alpha \oplus 2^\beta, N(t))\}]. \tag{27}$$

Now, in view of (9), if $N_\alpha(t)$, $N_\beta(t)$ are identically distributed and independent from the other noise processes,

$$E\{\mathrm{Wal}\,[i \oplus 2^\alpha \oplus 2^\beta, \bar{N}(t)]\}$$
$$= E\{\mathrm{Wal}\,[i, \bar{N}(t)]\} \cdot E\{(-1)^{i_\alpha N_\alpha(t) + i_\beta N_\beta(t)}\}. \quad (28)$$

Since exactly one out of $\{i_\alpha, i_\beta\}$ will be 1 if $i_\alpha \neq i_\beta$, and since $(-1)^{N_\alpha(t)}$ and $(-1)^{N_\beta(t)}$ have the same mean, it follows from (25) and (27) again that $\xi_Q(N) = \xi_S(N)$. Hence, we can make the following assertion.

A. Assertion II

For a given combinational circuit S, the noise-dependent error term $\xi_S(N)$ is 1) invariant under a complementation of its Boolean function and/or a complementation of any subset of the inputs, and 2) invariant under an interchange of inputs $x_\alpha(t)$ and $x_\beta(t)$ if the noise processes $\{N_\alpha(t), N_\beta(t)\}$ are independent of $\{N_i(t)\colon i \neq \alpha, \beta\}$ and the probabilities of $N_\alpha(t) = 1$ and of $N_\beta(t) = 1$ are equal.

V. ERROR FOR INDEPENDENT AND STATIONARY INPUTS

The noise-error formulas in (25) and (26) are quite involved, and it is difficult to interpret the dependence of $\xi_S(N)$ and $\xi_S(N, X)$ on the statistics of the noise and the signal. We shall now show that, when the signal and noise are independent and stationary processes, each identically distributed, $\xi_S(N)$ is expressible as a polynomial in the expected value of the noise and $\xi_S(N, X)$ as a polynomial in the expected values of the signal and the noise. Preparatory to the substantiation of this statement, we first need to establish a lemma.

A. Lemma (See Appendix II for Proof)

If $X_j \in \{0, 1\}$ are independent and identically distributed binary random variables ($0 \leqslant j < n$) with Prob $(X_j = 1) = \delta_X$, then

$$E\{\mathrm{Wal}\,(i, \bar{X})\} = (1 - 2\delta_X)^{Hm(i)} \quad (29)$$

where $Hm(i)$ denotes the Hamming weight of i:

$$Hm(i) = \sum_{j=0}^{n-1} i_j. \quad (30)$$

Under the assumption that the signal and noise are independent and stationary processes, each identically distributed, we obtain the following important simplified results from (25) and (26) immediately, with the aid of (29).

B. Assertion III

If $\{X_i(t), N_i(t)\colon 0 \leqslant i < n\}$ are independent and stationary with

$$\mathrm{Prob}\,[X_i(t) = 1] = E\{X_i\} = \delta_X \quad (31)$$

and

$$\mathrm{Prob}\,[N_i(t) = 1] = E\{N_i\} = \delta_N \quad (32)$$

then the component error expressions $\xi_S(N)$ and $\xi_S(N, X)$ in (25) and (26) for the Boolean function S represented in (10) can be simplified, respectively, to

$$\xi_S(N) = 2 \sum_{i \in D_n} b_i^2 [1 - p_N^{Hm(i)}] \quad (33)$$

and

$$\xi_S(N, X) = \sum_{i \neq j: i, j \in D_n} \sum b_i b_j [1 + p_N^{Hm(i \oplus j)}$$
$$- p_N^{Hm(i)} - p_N^{Hm(j)}]\, p_X^{Hm(i \oplus j)} \quad (34)$$

where

$$p_X = 1 - 2\delta_X \quad (35)$$

and

$$p_N = 1 - 2\delta_N \quad (36)$$

and D_n may be replaced by $D_n - \{0\}$. Note that $\xi_S(N)$ in (33) is a polynomial in the expected value of the noise and that $\xi_S(N, X)$ in (34) is a polynomial in the expected values of the signal and the noise. Both are relatively simple to compute and their sum can be plotted versus δ_N for different values of δ_X for a given combinational circuit.

VI. SPECIAL SITUATIONS

We now examine the behavior of the error for three special situations; namely, A) the low-noise case, B) the case of $\delta_N = 0.5$, and C) the unbiased-signal case.

A. The Low-Noise Case: $\delta_N \ll 1$

In such a case, we can use the approximation

$$p_N^k = (1 - 2\delta_N)^k \cong 1 - 2k\delta_N \quad (37)$$

and write (33) and (34) as

$$\xi_S(N) = 2\delta_N e_0 \quad (38)$$

and

$$\xi_S(N, X) = 2\delta_N \sum_{k=1}^{n} e_k p_X^k \quad (39)$$

where, for $0 \leqslant k \leqslant n$,

$$e_k = \sum_{\substack{i, j \in D_n \\ Hm(i \oplus j) = k}} \sum b_i b_j [Hm(i) + Hm(j) - Hm(i \oplus j)]. \quad (40)$$

Substituting (38) and (39) in (24), we have

$$\xi_S = 2\delta_N \sum_{k=0}^{n} e_k p_X^k. \quad (41)$$

Hence, we can conclude from (41) that, in a low-noise situation, the error ξ_S increases approximately linearly with δ_N, the constant of proportionality being a polynomial in p_X. This conclusion is exhibited in Figs. 2(a) and 3(a). Note that ξ_S is independent of the signal if, and only if, $e_k = 0$ for $0 < k \leqslant n$.

B. The Case of $\delta_N = 0.5$

This is a case of very high noise and, from (36), $p_N = 0$. We have

$$1 + p_N{}^{Hm(i \oplus j)} - p_N{}^{Hm(i)} - p_N{}^{Hm(j)}$$

$$= \begin{cases} 2, & \text{for } i = j \neq 0 \\ 1, & \text{for } i \neq j \neq 0 \\ 0, & \text{for } i \text{ or } j \text{ or both} = 0. \end{cases} \tag{42}$$

By substitution of (42) in (34) and then combining with (33), we obtain

$$\xi_S = \xi_S(N) + \xi_S(N, X) = \sum_{k=0}^{n} c_k p_X{}^k \tag{43}$$

where

$$c_0 = 2 \sum_{i \in D_n - \{0\}} b_i^2 \qquad (= \xi_S(N)) \tag{44}$$

and

$$c_k = \sum_{\substack{i,j \in D_n - \{0\} \\ Hm(i \oplus j) = k}} \sum b_i b_j, \qquad 1 \leqslant k \leqslant n. \tag{45}$$

Hence, ξ_S is a polynomial in p_X as in (41). ξ_S is independent of signal $\overline{X}(t)$ if, and only if, $c_k = 0$ for $1 \leqslant k \leqslant n$.

C. The Unbiased-Signal Case

In many situations the signal has no bias, or the environment is not known; it would be reasonable to assume $\delta_X = 0.5$ or $p_X = 0$. Hence $\xi_S(N, X) = 0$ and

$$\xi_S = \xi_S(N) = 2 \sum_{i \in D_n} b_i^2 [1 - p_N{}^{Hm(i)}]. \tag{46}$$

The error is then invariant over each equivalence class.

VII. NUMERICAL EXAMPLES AND ERROR CURVES

We shall now apply the method developed in the previous sections to determine the error at the output of two combinational circuits as a function of the expected values of the signal and of the noise at the input. Both the signal and the noise-component stochastic processes are assumed to be independent and stationary, each being identically distributed. The error ξ_S will be calculated and plotted versus δ_N for different values of δ_X. Two sets of error curves (one set for a low-noise and the other set for a high-noise situation) are presented for the Boolean function of each combinational circuit.

A. Circuit 1—A Three-Input Function: $S(\bar{x}) = x_2{}'x_1x_0{}' + x_2x_1{}'x_0{}' + x_2x_1$

For this Boolean function, $S(\bar{x}) = 1$ for $x \in \{2, 4, 6, 7\}$ and the vector $\bar{b}$, representing the coefficients b_i ($i = 0, 1, \cdots, 7$) of the Walsh-series expansion in (10), is found by (11) to be

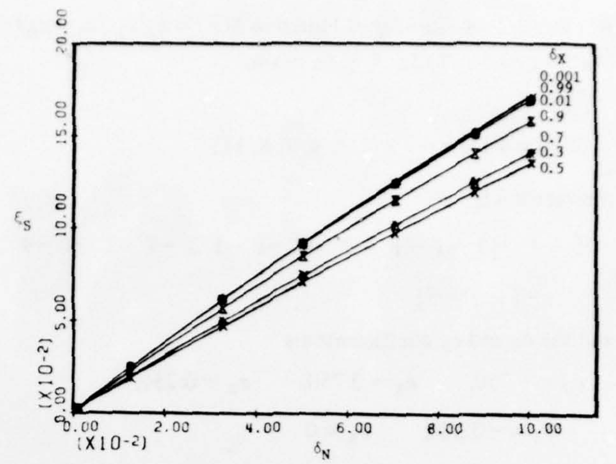
$$\bar{b} = 2^{-2}[2 \ \ 1 \ \ -1 \ \ 0 \ \ -1 \ \ 0 \ \ 0 \ \ -1].$$

In order to examine the behavior of the error curves in both the low-noise and the high-noise situations, the e_k coefficients in (40) and the c_k coefficients in (44) and (45) are computed.

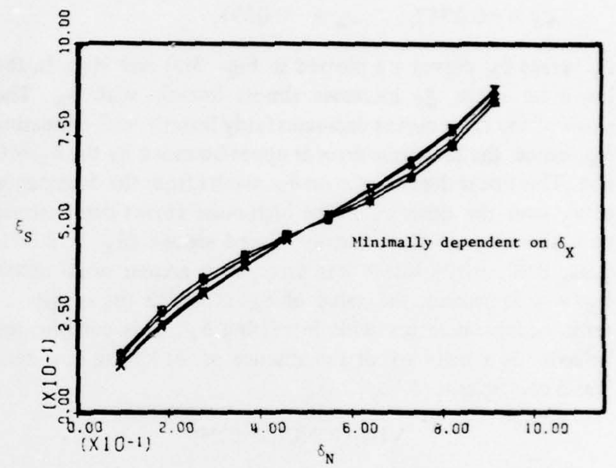
$$e_0 = 0.75, \qquad e_1 = e_3 = 0, \qquad e_2 = 0.25$$

$$c_0 = 0.50, \qquad c_1 = c_2 = c_3 = 0.$$

ξ_S versus δ_N curves are plotted in Figs. 2(a) and 2(b). It is seen that, in the low-noise range, ξ_S increases almost linearly with δ_N and is dependent on δ_X because e_2 is nonzero. In the entire high-noise range, ξ_S depends minimally on δ_X and becomes independent of δ_X at $\delta_N = 0.5$, in agreement with the c_k's being zero for $k \neq 0$. The error curves are monotonically increasing, and thus a maximum uncertainty ($\delta_N = 0.5$) does not cause a maximum error. The compactness of both the low-noise and the high-noise error curves is apparently due to the sparseness of the vector $\bar{b}$.



(a)



(b)

Fig. 2. (a) Low-noise error curve for circuit 1. (b) High-noise error curve for circuit 1.

B. Circuit 2–A Four-Input Function $S(\bar{x}) = x_3'x_2'(x_1 \oplus x_0) + (x_3 \oplus x_2)x_1'x_0' + (x_3' + x_3x_2')x_1x_0$

Here,

$$S(\bar{x}) = 1 \text{ for } x \in \{1, 2, 3, 4, 7, 8, 11\}$$

the vector $\bar{b}$ is

$$\bar{b} = 2^{-4}[7\ -1\ -1\ 3\ 3\ -1\ -1\ -1\ 3\ -1\ -1\ -1\ -1\ -1\ -1\ -5]$$

and the e_k and c_k coefficients are

$$e_0 = 1.250,\quad e_1 = 0.750,\quad e_2 = 0.250,\quad e_3 = -0.250,\quad e_4 = 0$$

$$c_0 = 0.492,\quad c_1 = 0.0312,\quad c_2 = -0.0156,\quad c_3 = -0.0312,\quad c_4 = -0.0391.$$

ξ_S versus δ_N curves are plotted in Figs. 3(a) and 3(b). In the low-noise range, ξ_S increases almost linearly with δ_N. The slope of the error curves decreases fairly linearly with increasing δ_X; hence, the low-noise error is upper-bounded by the $\delta_X = 0$ line. This linear dependence on δ_X results from the dominance of e_1 over the other e_k's. The high-noise curves demonstrate two phenomena: first, heavily biased signals (δ_X = 0.001, 0.01, 0.90, 0.99) suffer less error with greater noise above δ_N = 0.5; second, the value of δ_N at which the maximum error occurs increases with increasing δ_X. This complicated behavior is a hallmark of the absence of, or having few, zero Walsh coefficients (b_i's).

VIII. CONCLUSION

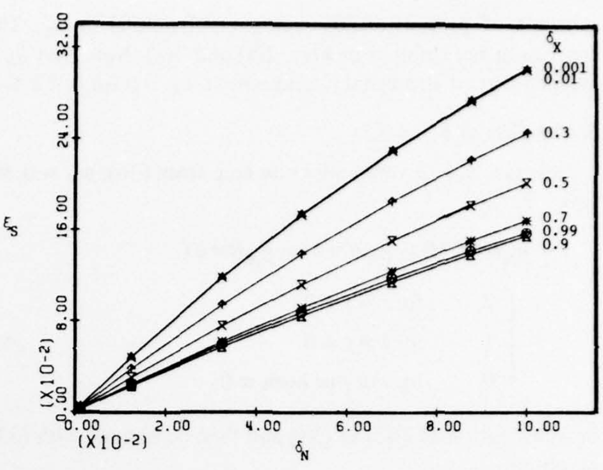
This paper presents a procedure for analyzing the stochastic behavior of digital circuits by the use of Walsh functions. In particular, the error at the output of a combinational circuit caused by noise is studied by defining a distance measure between the responses to the signal and to the noise-corrupted signal. By restricting the noise to be dyadically additive, which is perfectly reasonable, Walsh representation is used to obtain the error in terms of the input statistics. It is shown that the error can be expressed as the sum of two parts: one part depends only on noise statistics, and the other on both signal and noise. The former is invariant for equivalent Boolean functions, if the noise processes are independent and identically distributed. Under the more constrained condition of independent and stationary noise and signal processes, each identically distributed, the error is a polynomial function of the expected values of signal and of noise. In a low-noise situation, the error increases linearly with the expected value of the noise at the input. For unbiased signals, the error polynomial is invariant over each equivalence class. These properties are exhibited in two typical examples.

APPENDIX I

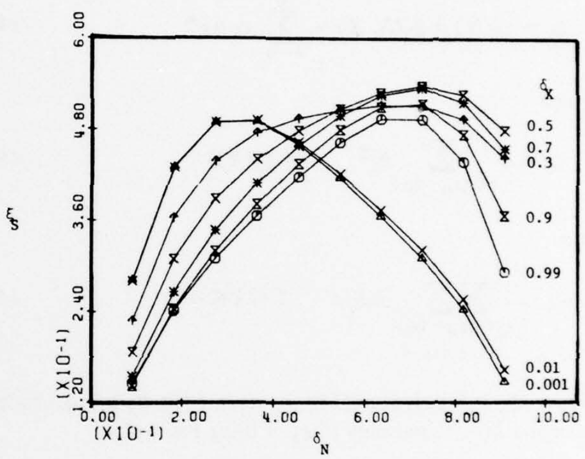
Proof of Assertion 1

a) By hypothesis,

$$Q(\bar{x}) = 1 - S(\bar{x}). \tag{A1}$$



(a)



(b)

Fig. 3. (a) Low-noise error curve for circuit 2. (b) High-noise error curve for circuit 2.

Using (10) and the fact that Wal $(0, \bar{x}) \equiv 1$, we have

$$Q(\bar{x}) = \text{Wal}\,(0, \bar{x}) - \sum_{i \in D_n} b_i \,\text{Wal}\,(i, \bar{x})$$

$$= (1 - b_0)\,\text{Wal}\,(0, \bar{x}) - \sum_{i \in D_n - \{0\}} b_i \,\text{Wal}\,(i, \bar{x}). \tag{A2}$$

Comparison of (A2) with (13) proves (14).

b) Complementing the m inputs changes $\bar{x}$ to $\bar{y} = \bar{x} \oplus \bar{h}$, with h given by (15a), which implies $h_{j_k} = 1$ for all k. As a result,

$$Q(\bar{x}) = S(\bar{y}) = S(\bar{x} \oplus \bar{h})$$

$$= \sum_{i \in D_n} b_i \,\text{Wal}\,(i, \bar{x} \oplus \bar{h})$$

$$= \sum_{i \in D_n} [b_i \,\text{Wal}\,(i, \bar{h})]\,\text{Wal}\,(i, \bar{x}) \tag{A3}$$

from which (15) follows directly.

c) Interchanging x_α and x_β changes $\bar{x}$ to $\bar{y}$ and

$$Q(\bar{x}) = S(\bar{y}) = \sum_{i_\alpha = i_\beta} b_i \operatorname{Wal}(i, \bar{y}) + \sum_{i_\alpha \neq i_\beta} b_i \operatorname{Wal}(i, \bar{y}). \tag{A4}$$

(c1) If $i_\alpha = i_\beta$, $\operatorname{Wal}(i, \bar{y}) = \operatorname{Wal}(i, \bar{x})$. Thus $q_i = b_i$.

(c2) If $i_\alpha \neq i_\beta$, then for $j = i \oplus 2^\alpha \oplus 2^\beta$, $j_\alpha \neq j_\beta$.

We have $\operatorname{Wal}(i, \bar{y}) = \operatorname{Wal}(j, \bar{x})$, and $\operatorname{Wal}(j, \bar{y}) = \operatorname{Wal}(i, \bar{x})$. Thus,

$$q_i = b_{i \oplus 2^\alpha \oplus 2^\beta}.$$

Equation (16) is therefore proved.

APPENDIX II

Proof of Lemma

We note from (6) that only those x_j's corresponding to $i_j = 1$ will have an effect on the value of $\operatorname{Wal}(i, \bar{x})$. If the X_j's are independent and identically distributed, any collection of that many X_j's will suffice for probability calculations for $\operatorname{Wal}(i, \bar{X})$. Thus

$$\begin{aligned} \operatorname{Prob}[\operatorname{Wal}(i, \bar{X}) = 1] &= \operatorname{Prob}\left[\left(\sum_{j=1}^{Hm(i)} X_j\right) \text{ is even}\right] \\ &= \sum_{\substack{w=0 \\ (w \text{ is even})}}^{Hm(i)} \operatorname{Prob}\left[\left(\sum_{j=1}^{Hm(i)} X_j\right) = w\right] \\ &= \sum_{\substack{w=0 \\ (w \text{ is even})}}^{Hm(i)} \binom{Hm(i)}{w} \delta_X{}^w (1 - \delta_X)^{Hm(i) - w}. \end{aligned} \tag{A5}$$

Similarly,

$$\begin{aligned} \operatorname{Prob}[\operatorname{Wal}(i, \bar{X}) = -1] &= \operatorname{Prob}\left[\left(\sum_{j=1}^{Hm(i)} X_j\right) \text{ is odd}\right] \\ &= \sum_{\substack{w=1 \\ (w \text{ is odd})}}^{Hm(i)} \binom{Hm(i)}{w} \delta_X{}^w (1 - \delta_X)^{Hm(i) - w}. \end{aligned} \tag{A6}$$

Combining (A5) and (A6), we have

$$\begin{aligned} E[\operatorname{Wal}(i, \bar{X})] &= \operatorname{Prob}[\operatorname{Wal}(i, \bar{X}) = 1] - \operatorname{Prob}[\operatorname{Wal}(i, \bar{X}) = -1] \\ &= \sum_{w=0}^{Hm(i)} \binom{Hm(i)}{w} (-\delta_X)^w (1 - \delta_X)^{Hm(i) - w} \\ &= (1 - 2\delta_X)^{Hm(i)} \end{aligned} \tag{A7}$$

which is (29), the lemma to be proved.

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